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A Certain Class of Cubic Surfaces treated by Quaternions.

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The general scalar equation of the third degree in an unknown vector ρ may be written in the form

$$H + Sa_1\rho b_1 + Sa\rho b\rho c + Sl'\rho m\rho n\rho p = 0,$$

where H is a known scalar quantity and a_1, b_1, a, b, &c., are known quaternions. But we have

1st. $Sa_1\rho b_1 = S(Sa_1 + Va_1)\,\rho\,(Sb_1 + Vb_1) = Sa_1 . S\rho Vb_1 + Sb_1 . S\rho Va_1 + S . \rho Vb_1 Va_1 .$
This may evidently be written in the form $S\delta\rho$, where δ is a known vector.

$$\text{2d.}\quad Sa\rho b\rho c = Sca\rho b\rho = Sa'\rho b\rho = (2Sa'Sb - Sa'b)\,\rho^2 + 2Sa'\rho Sb\rho .$$

This may be written $E\rho^2 + 2S\alpha\rho S\beta\rho$, where E is a known scalar quantity and α and β known vectors.

$$\begin{aligned}\text{3d.}\quad Sl'\rho m\rho n\rho p &= Spl'\rho m\rho n\rho = Sl\rho m\rho n\rho = Sl\rho m\rho\,(S\nu\rho + \rho Sn + V\nu\rho)\\ &= \rho^2 S\rho\,[\lambda\,(SmSn - S\mu\nu) + \mu\,(SnSl - S\nu\lambda) + \nu\,(SlSm - S\lambda\mu) - Sn . V\lambda\mu\\ &\qquad - Sl . V\mu\nu - Sm . V\nu\lambda] + 4S\lambda\rho S\mu\rho S\nu\rho,\end{aligned}$$

where λ, μ and ν are respectively equal to Vl, Vm and Vn.

This quantity may be written in the form $F\rho^2 S\eta\rho + 4S\lambda\rho S\mu\rho S\nu\rho$, where

$$\begin{aligned}F\eta = \lambda\,(SmSn - S\mu\nu) + \mu\,(SnSl - S\nu\lambda) + \nu\,(SlSm - S\lambda\mu)\\ - Sl . V\mu\nu - Sm . V\nu\lambda - Sn . V\lambda\mu.\end{aligned} \qquad \text{No. 1.}$$

Hence the general equation of the third degree may be written

$$H + S\delta\rho + E\rho^2 + 2S\alpha\rho S\beta\rho + F\rho^2 S\eta\rho + 4S\lambda\rho S\mu\rho S\nu\rho = 0.$$

If now we assume such relations of the constants and such an origin that the terms containing the first and second powers of ρ disappear for all values of ρ, though I have as yet been unable to discover any general method of classification, the above equation becomes

$$D + F\rho^2 S\eta\rho + 4S\lambda\rho Sm\rho S\nu\rho = 0, \qquad \text{No. 2.}$$

in which it must be remembered that η is a vector dependent upon the other constants. It may however be written in terms of the three known vectors λ, μ and ν in which case we have

$$D + A\rho^2 S\lambda\rho + B\rho^2 S\mu\rho + C\rho^2 S\nu\rho + 4S\lambda\rho S\mu\rho S\nu\rho = 0. \qquad \text{No. 3.}$$

This is the general equation of a class of surfaces of the third degree, which from analogy I call Central Cubics, whose peculiarities I propose to discuss somewhat briefly in the following pages.

Let us now define $\phi\rho f\rho$ as a vector function of ρ of the second degree. In case we have $S\lambda\phi\rho f\rho = S\rho\phi\lambda f\rho = S\rho\phi\rho f\lambda$ where λ is any vector, we may say that the function is self-conjugate. If however we have $S\lambda\phi\rho f\rho$ equal to $S\rho\phi\lambda f\rho$, but not to $S\rho\phi\rho fl$ we may say that the function is semi-self-conjugate. Having premised so much, let us assume a semi-self-conjugate vector function of ρ of the second degree

$$\phi\rho f\rho = A\rho S\lambda\rho + B\rho S\mu\rho + C\rho S\nu\rho + \frac{4}{3}\left[\lambda S\mu\rho S\nu\rho + \mu S\nu\rho S\lambda\rho + \nu S\lambda\rho S\mu\rho\right],$$

we may now write equation No. 3, $S\rho\phi\rho f\rho = D$, changing for convenience the sign of D. If D becomes equal to zero, we have

$$S\rho\phi\rho f\rho = 0, \qquad \text{No. 4.}$$

which is the equation of a cubical cone, and will be considered hereafter.

In case D is not zero, we may divide the constants in the above equation so that we may write

$$S\rho\phi\rho f\rho = 1, \qquad \text{No. 5.}$$

as the general equation of central surfaces of the third degree excluding cones.

If we differentiate No. 5 we have

$$Sd\rho\phi\rho f\rho + S\rho\phi d\rho f\rho + S\rho\phi\rho fd\rho = 0,$$

or, as the function is semi-self-conjugate, we may write

$$2Sd\rho\phi\rho f\rho + S\rho\phi\rho fd\rho = 0.$$

But $d\rho$ is in the direction of the variation of ρ at any instant. It is then in the direction of the tangent at the extremity of ρ. Now, if we consider ρ fixed and allow $d\rho$ to vary, we may write $(\omega - \rho)$, for $d\rho$ and the equation becomes, after reducing, $2S\omega\phi\rho f\rho + S\rho\phi\rho f\omega = 3$. This is the equation of the tangent plane.

If this plane pass through the point a, whose vector is α, we have $2S\alpha\phi\rho f\rho + S\rho\phi\rho f\alpha = 3$. This is the equation of the surface of the second degree, made by the contact of all possible tangent planes that pass through the point a, and may be called the polar quadric of the given point.

If we write the equation of the central cubic in the cyclic form we have

$$A\rho^2 S\lambda\rho + B\rho^2 S\mu\rho + C\rho^2 S\nu\rho + 4S\lambda\rho S\mu\rho S\nu\rho = 1,$$

where the vectors λ, μ and ν may be any vectors not inconsistent with the original supposition, by which I got rid of the first and second powers of ρ. It is, however, probable that these vectors are not restricted by this supposition, and in the rest of this discussion I have assumed them thus unrestricted. Of course such assumption is not warranted by strict mathematical logic, but the results seem so interesting and so consistent with other known mathematical truths, that I have felt myself warranted in my assumption, while waiting for the solution of what is apparently a very difficult problem, viz: the complete determination of ρ in the general vector function of the second degree. The very form of the equation proves to us the existence of three cyclic planes perpendicular to the three vectors λ, μ and ν.

To find the Radius of Curvature of any Normal Section of the Surface.

Differentiating equation No. 6, we have, assuming s as the independent variable,

$$S\,(2A\rho S\lambda\rho + A\rho^2\lambda + 2B\rho S\mu\rho + B\rho^2\mu + 2C\rho S\nu\rho + C\rho^2\nu + 4\lambda S\mu\rho S\nu\rho + 4\mu S\nu\rho S\lambda\rho \\ + 4\nu S\lambda\rho S\mu\rho)\,\rho' = 0\,,$$

where ρ' is the first differential coefficient of ρ with reference to s, its tensor, which can be any quantity, being assumed to be one.

Taking the second differential, we have

$$S\left[\frac{\begin{array}{c}2Ad\rho S\lambda\rho + 2A\lambda S\rho d\rho + 2Bd\rho S\mu\rho + 2B\mu S\rho d\rho + 2Cd\rho S\nu\rho + 2C\nu S\rho d\rho + 4\mu S\nu d\rho S\lambda\rho \\ + 4\mu S\nu\rho S\lambda d\rho + 4\nu S\lambda d\rho S\mu\rho + 4\nu S\lambda\rho S\mu d\rho + 4\lambda S\mu d\rho S\nu\rho + 4\lambda S\mu\rho S\nu d\rho\end{array}}{ds}\right]\rho' + S\beta\rho'' = 0,$$

where β is put for convenience for the normal vector. But ρ'' is perpendicular to β, hence we have, after some reduction, the following equation to determine the radius of curvature—

$$T\rho'' = \frac{1}{\beta}\,S\left[\frac{\begin{array}{c}2Ad\rho S\lambda\rho + 2A\lambda S\rho d\rho + 2Bd\rho S\mu\rho + 2B\mu S\rho d\rho + 2Cd\rho S\nu\rho + 2C\nu S\rho d\rho + 4\lambda S\mu d\rho S\nu\rho \\ + 4\lambda S\mu\rho S\nu d\rho + 4\mu S\nu d\rho S\lambda\rho + 4\mu S\nu\rho S\lambda d\rho + 4\nu S\lambda d\rho S\mu\rho + 4\nu S\lambda\rho S\mu d\rho\end{array}}{d\rho}\right],$$

where $T\rho''$ is the reciprocal of the radius of absolute curvature. But we know that $d\rho$ is a vector in the tangent plane, and because the above expression is independent of $Td\rho$, we can assume any vector t, whose tensor is equal to one that lies in the tangent plane, to be equal to $d\rho$. Substituting this, we have

$$T\rho'' = -\frac{1}{\beta} S\rho\,[2A\lambda + 2AtS\lambda t + 2B\mu + 2BtS\mu t + 2C\nu + 2CtS\nu t + 8\mu S\lambda tS\nu t + 8\nu S\lambda tS\mu t + 8\lambda S\mu tS\nu t],$$

which is the reciprocal of the radius of absolute curvature of any normal section in the direction of t.

If the right hand side of the equation reduces to zero for any given value of ρ, there is either a point of double inflexion, or the tensor of ρ becoming infinite, the point is at infinity, where the asymptotic line is tangent to the curve.

If now we assume t to be a vector in a plane passing through the origin perpendicular to λ, ρ will also be a vector in that plane; and the right hand member of the above equation becomes

$$-\frac{1}{\beta} S\rho\,(2B\mu + 2BtS\mu t + 2C\nu + 2CtS\nu t).$$

In order that this quantity may be equal to zero, ρ must be perpendicular to the vector

$$2B\mu + 2C\nu + t\,(2BS\mu t + 2CS\nu t),$$

and then represents probably the point of double inflation of the curvature of the section. Or we may suppose t to be an asymptotic line; in this case, ρ and t become parallel to each other, when, if $S\,(\mu 2B + \nu 2C)\,\rho = 0$, then $S\,(\mu 2B + \nu 2C)\;t = 0$, and the whole quantity becomes equal to zero. Hence, at that point, as befere, the radius of curvature is infinite. These properties of the curve will be studied more in detail hereafter, and are only referred to at present as illustrations of the use of quaternions in studying the radius of curvature.

If, in equation No. 6, the constant term disappears, the equation represents a cubical cone, as the tensor of ρ may be any quantity. If now we suppose this surface to be cut by the sphere of $\rho^2 = -1$, we have, as the equation of the Spherical Conic of the third degree,

$$4S\lambda\rho S\mu\rho S\nu\rho = AS\lambda\rho + BS\mu\rho + CS\nu\rho,$$

in which $T\rho = 1$. Hence to construct a spherical conic we have the following rule: Assuming any three points on a sphere not in the same plane, then if the continued product of the cosines of the angles which these points make with the variable point, is equal to the sum of the cosines of the angles that the variable point makes with the three given points each multiplied by some constant quantity, then the locus of the variable point is a spherical cubic conic.

If the general surface is cut by one of the cyclic planes, as for example $S\lambda\rho = 0$, we have a cubic curve whose equation may be written

$$\rho^2 S\,(B\mu + C\nu)\,\rho = 1.$$

This curve may be called by analogy a cubical circle. It has for an asymptotic line, a vector perpendicular to $B\mu + C\nu$.

If we cut the surface by the plane $S\,(B\mu + C\nu)\,\rho = 0$, we have

$$A\rho^2 S\lambda\rho + 4S\lambda\rho S\mu\rho S\nu\rho = 1,$$

a curve which has for an asymptotic line a vector perpendicular to λ. This equation represents a curve, that may be called a cubical ellipse.

If we cut the surface by the plane $S\,(A\lambda + B\mu + C\nu)\,\rho = 0$, we have $S\lambda\rho S\mu\rho S\nu\rho = 1$, omitting the 4 for convenience. This is a curve which has asymptotic lines in three directions perpendicular respectively to λ, μ and ν, and may be called the cubical hyperbola.

Thus far we have considered the central surface only under its most general form. If now we consider certain relations to exist between the given constants, several families of surfaces readily suggest themselves.

First. If in equation No. 6 we make λ, μ and ν parallel to each other, the equation may be written

$$D\rho^2 S\lambda\rho + S^3\lambda\rho = C, \qquad \text{No. 7.}$$

where λ is a unit vector. This surface has as its limit the plane $S\lambda\rho = 0$; and as its central asymptotic line the vector $-\lambda$ prolonged, in either of which directions $T\rho = \infty$.

If we cut this surface by any plane $S\lambda\rho = -b$, we have $T\rho = \sqrt{\frac{b^3 + C}{bD}}$, the equation of a circle. Equation No. 7 represents a surface of revolution around the vector λ, and may perhaps be called a cubic cylinder.

To find that transverse plane which shall so cut the surface as to render $T\rho$ a minimum we have, considering b the independent variable and taking the first differential,

$$\frac{dT\rho}{db} = \frac{2Db^3 - DC}{2b^2D^2} \times \sqrt{\frac{bD}{b^3 + C}}.$$

In order that $T\rho$ should be a minimum, the equation must be equal to zero, which gives, as the condition required,

$$b = \sqrt[3]{\frac{C}{2}}.$$

If, in equation No. 7, we suppose D to become equal to one, its relation to an ordinary circular cylinder becomes at once apparent. We have then

$$\rho^2 S\lambda\rho + S^3\lambda\rho = C.$$

This may be changed, according to a well-known formula of quaternions, into $\sqrt{-S\lambda\rho} \,.\, TV\lambda\rho = \sqrt{C}$, and its relation to the circular cylinder becomes at once apparent, the equation of the latter being $TV\lambda\rho = A$.

Second. If, in equation No. 1, we make Sl, Sm and Sn equal to zero, and ν a vector parallel to $V\lambda\mu$, the general equation reduces to the comparatively simple form

$$A\rho^2 S\nu\rho + S\lambda\rho S\mu\rho S\nu\rho = C.$$

This equation may be written $S\rho\phi\rho S\nu\rho = C$, where $\phi\rho$ is a self-conjugate linear and vector function of ρ equal to $A\rho + \frac{1}{2}\,(\lambda S\mu\rho + \mu S\lambda\rho)$. This equation represents a family of surfaces of the third degree very closely related to the quadric surfaces. If C becomes zero, the equation is satisfied by any vector in the plane $S\nu\rho = 0$, and in the cone $S\rho\phi\rho = 0$. Hence it degenerates into a quadric cone and a plane at the apex. If C is not zero, we may divide the constants, so as to write the equation

$$S\rho\phi\rho S\nu\rho = 1, \qquad \text{No. 8.}$$

which is the general equation of ellipsoids and hyperboloids of this family of surfaces. Writing this equation in what Hamilton calls the rectangular form we have $cS^2 i\rho Sj\rho + c_1 S^3 j\rho + c_{11} S^2 k\rho Sj\rho = 1$, which is either an ellipsoid or hyperboloid according as c, c_1 and c_{11} are all of the same sign, or part positive and part negative. It ought perhaps to be noticed that in case the signs of all the terms are negative, the surface does not become imaginary as in the case of the corresponding quadric, but is a real surface inasmuch as the cube root of a negative quantity is a real quantity.

If we assume the order of inequality $c < c_1 < c_{11}$ and make c_1 equal to zero, the equation becomes

$$cS^2 i\rho Sj\rho + c_{11} S^2 k\rho Sj\rho = 1, \qquad \text{No. 9.}$$

the equation of a hyperbolic cubic cylinder, as c is necessarily a negative quantity. This surface is cut by any plane $Sj\rho = b$ in an ordinary hyperbola.

Now, it can be easily shown that the assumption that c_1 equals zero in this family of surfaces, obliges λ and μ, if real vectors, to be at right angles to each other. This being the case, if we wish to change equation No. 9 to the cyclic form, we may write the very simple equation $S\lambda\rho S\mu\rho S\nu\rho = 1$ as that of a particular kind of a cubic hyperbolic cylinder in which λ, μ and ν are mutually perpendicular. If we wish to express this surface in Cartesian co-ordinates we may assume, as the axes of x, y and z, lines parallel respectively to λ, μ and ν and write the very simple form $xyz = 1$.

This surface is only a particular case of the general cubic hyperbolic cylinder, in which the directions of λ, μ and ν are unlimited, the derivation of which will be shown hereafter.

Third. The fact that this family of surfaces may be expressed in terms of a variable subsidiary quadric and variable parallel planes, suggests that a similar principle may be applied to the general central cubic surface. Thus we may write the latter as follows:

$$\left(A\rho^2 + \frac{4}{3}S\mu\rho S\nu\rho\right) S\lambda\rho + \left(B\rho^2 + \frac{4}{3}S\nu\rho S\lambda\rho\right) S\mu\rho + \left(C\rho^2 + \frac{4}{3}S\lambda\rho S\mu\rho\right) S\nu\rho = 1\,.$$

If we put, for the quantities in parenthesis, $S\rho\phi\rho$, $S\rho\chi\rho$ and $S\rho\psi\rho$, where $\phi\rho$, $\chi\rho$ and $\psi\rho$ are three mutually connected linear and vector self-conjugate functions of ρ, the equation becomes

$$S\rho\phi\rho S\lambda\rho + S\rho\chi\rho S\mu\rho + S\rho\psi\rho S\nu\rho = 1\,,$$

and the dependence of the general cubic surface on three connected subsidiary quadrics becomes at once apparent. If now we assume three sets of rectangular vectors, i, j, k; i_1, j_1, k_1, and i_{11}, j_{11}, k_{11}, we may write the equation

$$\begin{aligned}(aS^2i\rho + a_1S^2j\rho + a_{11}S^2k\rho)\, S\lambda\rho + (bS^2i_1\rho + b_1S^2j_1\rho + b_{11}S^2k_1\rho)\, S\mu\rho \\ + (cS^2i_{11}\rho + c_1S^2j_{11}\rho + c_{11}8^2k_{11}\rho)\, S\nu\rho = 1. \qquad \text{No. 10.}\end{aligned}$$

In this equation we have the following equalities:

$$\begin{array}{llllll}
i = & U(\mu T\nu - \nu T\mu) & i_1 = & U(\lambda T\nu - \nu T\lambda) & i_{11} = & U(\lambda T\mu - \mu T\lambda\,, \\
j = & UV\mu\nu\,, & j_1 = & UV\lambda\nu\,, & j_{11} = & UV\lambda\mu\,, \\
k = & U(\mu T\nu + \nu T\mu)\,, & k_1 = & U(\lambda T\nu + \nu T\lambda)\,, & k_{11} = & U(\lambda T\mu + \mu T\lambda)\,, \\
a = & -(A + S\mu\nu) - T\mu\nu\,, & b = & -(B + S\lambda\nu) - T\lambda\nu\,, & c = & -(C + S\lambda\mu) - T\lambda\mu\,, \\
a_1 = & -(A + S\mu\nu) + S\mu\nu\,, & b_1 = & -(B + S\lambda\nu) + S\lambda\nu\,, & c_1 = & -(C + S\lambda\mu) + S\lambda\mu\,, \\
a_{11} = & -(A + S\mu\nu) + T\mu\nu\,, & b_{11} = & -(B + S\lambda\nu) + T\lambda\nu\,, & c_{11} = & -(C + S\lambda\mu) + T\lambda\mu\,,
\end{array}$$

where we assume $a < a_1 < a_{11}$, $b < b_1 < b_{11}$ and $c < c_1 < c_{11}$.

This surface may be considered as one, two or threefold ellipsoidal, or one, two or threefold hyperboloidal, according as the signs of each term in one, two or all of the subsidiary quadrics are the same or different.

At this point it becomes necessary to express A, B and C of equation No. 3 in terms of Sl, Sm, Sn and $S\lambda\mu$, $S\mu\nu$ and $S\nu\lambda$ of the general equation. We have as before

$$F\eta = \lambda\,(SmSn - S\mu\nu) + \mu\,(SnSl - S\nu\lambda) + \nu\,(SlSm - S\lambda\mu)$$

$$- Sl\,.\,V\mu\nu - Sm\,.\,V\nu\lambda - Sn\,.\,V\lambda\mu$$

$$= \lambda\left[+ SmSn - S\mu\nu + \frac{Sn\,(S\lambda\mu S\mu\nu - \mu^2 S\nu\lambda) + Sm\,(S\mu\nu S\nu\lambda - \nu^2 S\mu\lambda) + Sl\,(\mu^2\nu^2 - S^2\mu\nu)}{S\lambda\mu\nu}\right]$$

$$+ \mu\left[SnSl - S\nu\lambda + \frac{Sn\,(S\nu\lambda S\mu\nu - \lambda^2 S\mu\nu) + Sm\,(\nu^2\lambda^2 - S^2\nu\lambda) + Sl\,(S\mu\nu S\nu\lambda - \nu^2 S\lambda\mu)}{S\lambda\mu\nu}\right]$$

$$+ \nu\left[SlSm - S\lambda\mu + \frac{Sn\,(\lambda^2\mu^2 - S^2\lambda\mu) + Sm\,(S\mu\nu S\nu\lambda - \lambda^2 S\mu\nu) - Sl\,(S\lambda\mu S\mu\nu - \mu^2 S\nu\lambda)}{S\lambda\mu\nu}\right]$$

Here the quantities in brackets are respectively equal to A, B and C of equation No. 3. Now it is evident that whatever are the vectors λ, μ, ν we can assume such a value of Sm as will render A equal to zero. In this case a_1 in equation No. 10 becomes zero, and the surface is a onefold cubic hyperbolic cylinder. In the cyclical form its equation would be

$$B\rho^2 S\mu\rho + C\rho^2 S\nu\rho + 4S\lambda\rho S\mu\rho S\nu\rho = 1. \qquad \text{No. 11.}$$

Again having assumed such a value of Sm as will render A equal to zero, we can easily determine a value of Sn which will render B also equal to zero. In this case a_1 and b_1 both become zero, and the surface is a twofold hyperbolic cubic cylinder and its equation in the cyclic form would be

$$C\rho^2 S\nu\rho + 4S\lambda\rho S\mu\rho S\nu\rho = 1. \qquad \text{No. 12.}$$

Lastly, if we can discover real values for Sl, Sm and Sn such that A, B and C all become equal to zero, a_1, b_1 and c_1 are all zero and the equation represents a threefold hyperbolic cubic cylinder. To do this let us put for convenience

$$\begin{array}{llll} S\lambda\mu = -a, & S\lambda\mu\nu = -d, & T\nu^2 = C, & Sn = z, \\ S\mu\nu = -b, & T\lambda^2 = A, & Sl = x, & \\ S\nu\lambda = -c, & T\mu^2 = B, & Sm = y. & \end{array}$$

We now have the following equations in which to determine x, y and z.

$$\begin{aligned} &- bd - dyz + zab - zBc + ybc - yCa - xb^2 + xBC = 0, \\ &- cd - dzx + zbc - zAb - yc^2 + yCA + xbc - xCa = 0, \\ &- ad - dxy - za^2 + zAB + ybc - yAb + xab - xBc = 0. \end{aligned}$$

Combining these three equations, we discover single definite values for x, y and z. Hence we have single definite values of Sl, Sm and Sn, which will render A, B and C of equation No. 3 equal to zero, and the problem is solved. The cyclic equation of this surface may be written $S\lambda\mu S\mu\rho S\nu\rho = 1$, where λ, μ and ν may be any non-parallel vectors.

In the preceding pages I have attempted to classify very briefly some of the various families of the central cubic surface, and have suggested only just enough of their properties to enable one partially to conceive of their several shapes, using only the more simple of the quaternion methods. If I continue the subject, I will endeavor, in another paper, to examine more carefully a few of the surfaces here enumerated.

NOTE.—In naming the surfaces described in the preceding pages, it has seemed best to me to follow their algebraical, rather than their geometrical, analogies with the quadric surfaces.

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